

SEMI-INVARIANTS OF 2-REPRESENTATIONS OF QUIVERS

STANISLAV FEDOTOV

ABSTRACT. In this work we obtain a version of the Procesi-Razmyslov Theorem for the algebra of semi-invariants of representations of an arbitrary quiver with dimension vector $(2, 2, \dots, 2)$.

1. INTRODUCTION

We work over a base field \mathbb{k} of characteristic zero. A quiver Q is a directed graph, determined by two finite sets Q_0 (the set of “vertices”) and Q_1 (the set of “arrows”) with two maps $h, t : Q_1 \rightarrow Q_0$ which indicate the vertices at the head and tail of each arrow. A representation (W, φ) of Q consists of a collection of finite dimensional \mathbb{k} -vector spaces W_v , for each $v \in Q_0$, together with linear maps $\varphi_a : W_{ta} \rightarrow W_{ha}$, for each $a \in Q_1$. The dimension vector $\alpha \in \mathbb{Z}^{Q_0}$ of such a representation is given by $\alpha_v = \dim_{\mathbb{k}} W_v$. A morphism $f : (W_v, \varphi_a) \rightarrow (U_v, \psi_a)$ of representations consists of linear maps $f_v : W_v \rightarrow U_v$, for each $v \in Q_0$, such that $f_{ha}\varphi_a = \psi_a f_{ta}$, for each $a \in Q_1$. Evidently, it is an isomorphism if and only if each f_v is.

Having chosen vector spaces W_v of dimension α_v , the isomorphism classes of representations of Q with dimension vector α are in natural one-to-one correspondence with the orbits of the group

$$GL(\alpha) := \prod_{v \in Q_0} GL(W_v)$$

in the representation space

$$\text{Rep}(Q, \alpha) := \bigoplus_{a \in Q_1} \text{Hom}(W_{ta}, W_{ha}).$$

This action is given by $(g \cdot \varphi)_a = g_{ha}\varphi_a g_{ta}^{-1}$, where $g = (g_v)_{v \in Q_0} \in GL(\alpha)$. Note that the one-parameter subgroup $\Delta = \{(tE, \dots, tE)\}$ acts trivially.

One can also consider the action of the smaller group $SL(\alpha) = \prod_{v \in Q_0} SL(W_v)$ on $\text{Rep}(Q, \alpha)$; $SL(\alpha)$ -invariant functions on $\text{Rep}(Q, \alpha)$ are usually called semi-invariants. In this work we study generators of the algebra $\mathbb{k}[\text{Rep}(Q, \alpha)]^{SL(\alpha)}$. Recall that generators of $\mathbb{k}[\text{Rep}(Q, \alpha)]^{GL(\alpha)}$ are given by the Procesi-Razmyslov Theorem.

Theorem 1. [5, Theorem 3.4], [6] *For an arbitrary quiver Q and dimension vector α the algebra $\mathbb{k}[\text{Rep}(Q, \alpha)]^{GL(\alpha)}$ is generated by the traces of oriented cycles of length not greater than $(\sum_v \alpha_v)^2$. All relations among them can be deduced from Cayley-Hamilton polynomials.*

As for the algebra $\mathbb{k}[\text{Rep}(Q, \alpha)]^{SL(\alpha)}$, we only have several descriptions of its spanning sets; the main approaches are presented in [3], [2] and [7]. In this work we describe a generating set for the algebra of 2-representations of an arbitrary quiver, i.e., of representations with dimension vector $(2, 2, \dots, 2)$, in the spirit of the Procesi-Razmyslov Theorem.

For a matrix A we will, as usually, denote by \hat{A} its adjoint matrix, i.e., the matrix consisting of cofactors to the elements of A^T . If A_F is the matrix of a linear map $F : U \rightarrow V$, then it is convenient to assume that its adjoint matrix defines a linear map from V to U .

Consider a quiver Q . Let $Q_0 = \{1, \dots, n\}$ and $Q_1 = \{a_1, \dots, a_s\}$. To Q we associate the quiver \tilde{Q} with $\tilde{Q}_0 = Q_0$ and $\tilde{Q}_1 = \{a_1, \dots, a_s\} \cup \{b_1, \dots, b_s\}$, where $hb_i = ta_i$, $tb_i = ha_i$. For each representation (W, φ) of Q , consider the associated representation of \tilde{Q} with the same spaces W_v and maps φ_{a_i} and

2000 *Mathematics Subject Classification.* Primary 16G20; Secondary 14L30.
Supported by grant RFFI 09-01-90416 - Ukr-f-a.

$\varphi_{b_i} = \widehat{\varphi}_{a_i}$. By a *route* in Q we mean an oriented cycle in \widetilde{Q} . For example, any cycle in Q is a route. We say that a route is *simple*, if no edge appears twice in the corresponding oriented cycle. The *trace of a route* is the trace of the corresponding cycle in \widetilde{Q} in the associated representation.

Theorem 2. *For a quiver Q and a dimension vector $\alpha = (2, 2, \dots, 2)$ the algebra $\mathbb{k}[\text{Rep}(Q, \alpha)]^{SL(\alpha)}$ is generated by the traces of simple routes.*

Example 1. Let Q be the quiver $1 \xrightleftharpoons[b]{a} 2$. Denote by $|X|$ the determinant of a matrix X . By Theorem 2 the algebra $\mathbb{k}[\text{Rep}(Q, (2, 2))]^{SL(2) \times SL(2)}$ is generated by $\text{tr } \varphi_a \widehat{\varphi}_a = 2|\varphi_a|$, $\text{tr } \varphi_b \widehat{\varphi}_b = 2|\varphi_b|$ and $\text{tr } \varphi_a \widehat{\varphi}_b$ (note that $\text{tr } \varphi_b \widehat{\varphi}_a = \text{tr } \widehat{\varphi}_a \varphi_b = \text{tr } \varphi_a \widehat{\varphi}_b$). So, if $\varphi_a = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$, $\varphi_b = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$, then $\mathbb{k}[\text{Rep}(Q, (2, 2))]^{SL(2) \times SL(2)} = \mathbb{k}[2x_{11}x_{22} - 2x_{12}x_{21}, 2y_{11}y_{22} - 2y_{12}y_{21}, x_{11}y_{22} - x_{12}y_{21} - x_{21}y_{12} + x_{22}y_{11}]$.

Our arguments use the description of a spanning set of the algebra $\mathbb{k}[\text{Rep}(Q, \alpha)]^{SL(\alpha)}$ devised by Domokos and Zubkov [3]; we briefly recall it in Section 2. In Sections 3 and 4 we obtain a formula expressing the determinant of a 2-block matrix as a polynomial in the traces of associated routes. In Section 5 we prove Theorem 2.

The author hopes to use the information received about the basic semi-invariants to generalize King's construction [4] in the way similar to [1].

The author thanks I.V. Arzhantsev for the idea of the work and useful discussions.

2. THE DOMOKOS-ZUBKOV THEOREM

Let us recall the results of [3]. Let $Q_0 = \{1, \dots, n\}$. Fix a dimension vector α and two tuples $\overline{i} = (i_1, \dots, i_k)$ and $\overline{j} = (j_1, \dots, j_l)$ of integers from 1 to n (possibly repeating) such that $(\alpha_{i_1} + \dots + \alpha_{i_k}) = (\alpha_{j_1} + \dots + \alpha_{j_l})$, and consider all possible matrices of size $(\alpha_{i_1} + \dots + \alpha_{i_k}) \times (\alpha_{j_1} + \dots + \alpha_{j_l})$ of form

$$A_{\overline{i}\overline{j}} := \begin{matrix} & i_1 & \dots & i_s & \dots & i_k \\ \begin{matrix} j_1 \\ \vdots \\ j_r \\ \vdots \\ j_l \end{matrix} & \begin{pmatrix} y_{11}F_{11} & \dots & y_{1s}F_{1s} & \dots & y_{1k}F_{1k} \\ \vdots & & \vdots & & \vdots \\ y_{r1}F_{r1} & \dots & y_{rs}F_{rs} & \dots & y_{rk}F_{rk} \\ \vdots & & \vdots & & \vdots \\ y_{l1}F_{l1} & \dots & y_{ls}F_{ls} & \dots & y_{lk}F_{lk} \end{pmatrix} \end{matrix}, \quad (3)$$

where y_{rs} are formal variables, and each matrix F_{rs} may be either 0, or the matrix of a map, that corresponds to an arrow going from the i_s -th vertex to the j_r -th one, or an identity matrix if $i_s = j_r$. For a fixed representation (W, φ) a matrix of form (3) defines a map from $W_{i_1} \oplus \dots \oplus W_{i_k}$ to $W_{j_1} \oplus \dots \oplus W_{j_l}$. Moreover, its determinant is a polynomial in variables y_{rs} with $SL(\alpha)$ -invariant coefficients: $|A_{\overline{i}\overline{j}}| = \sum_{\overline{\mu}} y^{\overline{\mu}} h_{\overline{\mu}}(x_{pq}^{rs})$, where $\overline{\mu} = (\mu_{ij})_{i,j=1}^{k,l}$ are multidegrees, and x_{pq}^{rs} are matrix elements of the matrices F_{rs} .

Theorem 3. [3, Thm. 4.1] *The algebra $\mathbb{k}[\text{Rep}(Q, \alpha)]^{SL(\alpha)}$ is spanned by semi-invariants $h_{\overline{\mu}}(x_{pq}^{rs})$.*

Thus all basic semi-invariants of Q are given by coefficients of monomials in variables y_{rs} in determinants of block matrices of form (3). We will describe them precisely for $\alpha = (2, 2, \dots, 2)$.

3. BLOCK MATRICES AND ASSOCIATED ROUTES

For a matrix $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ its adjoint matrix \widehat{X} equals $\begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix}$. Recall that for $X, Y \in \text{Mat}_{2 \times 2}(\mathbb{k})$ we have $\text{tr } X\widehat{Y} = \text{tr } \widehat{X}Y = \text{tr } X \text{tr } Y - \text{tr } XY$, $\text{tr } \widehat{X} = \text{tr } X$ and $\widehat{X}\widehat{Y} = \widehat{Y}\widehat{X}$.

Lemma 1. *Let $X, Y, Z \in \text{Mat}_{2 \times 2}(\mathbb{k})$. Then*

- (1) $\text{tr } X^2 Y = \text{tr } X \text{tr } XY - |X| \text{tr } Y$;
- (2) $\text{tr } XYXZ = \text{tr } XY \text{tr } XZ - |X| \text{tr } Y\widehat{Z}$.

Proof. It suffices to prove a polynomial identity for the elements of a Zariski open subset. Hence, we may assume that all matrices are invertible. By the Cayley-Hamilton Theorem for two by two matrices we have

$$X^2 - (\operatorname{tr} X)X + |X|E = 0.$$

Multiplying this equality by Y and taking the trace of the product received, we get (1). If Y is invertible, it follows that

$$\begin{aligned} \operatorname{tr} XYXZ &= \operatorname{tr} ((XY)^2 Y^{-1} Z) = \operatorname{tr} XY \operatorname{tr} (XY \cdot Y^{-1} Z) - |XY| \operatorname{tr} Y^{-1} Z = \\ &= \operatorname{tr} XY \operatorname{tr} (XZ) - |X| \operatorname{tr} \hat{Y} Z. \end{aligned} \quad \square$$

Consider a matrix $Z \in \operatorname{Mat}_{2k \times 2k}(\mathbb{k})$ divided into blocks X_{ij} of size 2×2 . To Z we associate a quiver Γ with $\Gamma_0 = \{1, \dots, k\} \cup \{-1, \dots, -k\}$, $\Gamma_1 = \{a_{ij} \mid i, j = 1, \dots, k\}$, $ha_{ij} = i$ and $ta_{ij} = -j$. A route in Γ defines a sequence of matrices: if an arrow goes from $-j$ to i , we take X_{ij} , and if it goes from k to $-l$, we take \hat{X}_{kl} . Thus constructed sequences we call *routes in Z* . The *adjoint* for a given route $P = (X_1, \dots, X_N)$ is $\hat{P} = (\hat{X}_N, \dots, \hat{X}_1)$. Observe that any cyclic permutation of factors as well as taking the adjoint route does not change the trace of the product of matrices along a route. We say that two routes are *equivalent* if there is such a transformation turning one of them into another. We claim that the determinant of Z is a polynomial in traces of its routes. To prove this we need to introduce the following construction.

Construction. The determinant of Z equals $|Z| = \sum_{\sigma \in S_{2k}} (-1)^\sigma z_{1\sigma(1)} \cdot z_{2\sigma(2)} \cdot \dots \cdot z_{2k\sigma(2k)}$. To each summand $z_\sigma := z_{1\sigma(1)} \cdot z_{2\sigma(2)} \cdot \dots \cdot z_{2k\sigma(2k)}$ we assign the *associated route set* \mathcal{P}_σ constructed as follows. Let $z_{1\sigma(1)}$ be in a block X_{1s_1} .

1a) if $z_{2\sigma(2)}$ is in the same block as $z_{1\sigma(1)}$, then add to \mathcal{P}_σ the route $(\hat{X}_{1s_1}, X_{1s_1})$ and consider the submatrix $Z_{\sigma(1)\sigma(2)}^{12}$ accessory to X_{1s_1} ;

1b) let $z_{2\sigma(2)}$ be in some other X_{1s_2} . In this case we consider the factor $z_{t_3\sigma(t_3)}$ that lies in the same block column as $z_{2\sigma(2)}$, i.e., in some $X_{r_3s_2}$;

2) consider the factor $z_{t_4\sigma(t_4)}$ lying in the same block row as the previous one, i.e., in some $X_{r_3s_4}$;

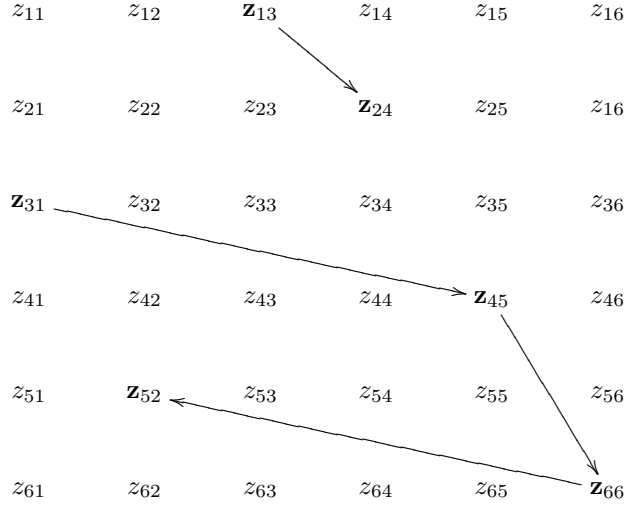
3) continue this process as long as it is possible, alternating horizontal and vertical shifts;

4) there comes a moment when we can not make another, some N -th shift. If it were the time to move vertically (respectively horizontally), then we have arrived to some p -th block column (respectively to a p -th block row) for the second time. But in each block column (except the s_1 -th) and in each block row we have already chosen two blocks (and hence, two factors of z_σ), so it is impossible to get there again. Therefore just before the algorithm failed we had arrived to the s_1 -th block column. It follows that N is even (because the last shift had been horizontal) and moreover, the last considered block was some $X_{r_N s_1}$;

5) consider the route $P = (\hat{X}_{1s_1}, X_{1s_2}, \hat{X}_{r_3s_2}, X_{r_3s_4}, \dots, X_{r_N s_1})$ (all odd factors are adjoint). We add it to \mathcal{P}_σ and pass to the submatrix matrix $Z_{\bar{P}}$ of size $(k - N) \times (k - N)$ containing all the factors of z_σ that we have not used yet.

Note that the construction is not unique: we may take various starting elements and choose one of two possible directions of circuit. These transformations correspond to cyclic permutations of the routes and/or taking the adjoint routes, hence not changing their equivalence classes.

Example 2. We illustrate the construction for the block matrix $Z = (X_{rs})_{r,s=1}^3$ and the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 5 & 2 & 6 \end{pmatrix}$:



The summand z_σ equals $z_{13}z_{24}z_{31}z_{45}z_{56}z_{61}$. Starting at z_{13} , we next take the factor z_{24} . Since both of them are in the same block X_{12} , we add to \mathcal{P}_σ the route (\hat{X}_{12}, X_{12}) and consider the submatrix $Z_{34}^{12} = \begin{pmatrix} X_{21} & X_{23} \\ X_{31} & X_{33} \end{pmatrix}$. Here we start at z_{31} , then take z_{45} . This element is in the second block column of Z_{34}^{12} ; we should take another factor lying in this column, that is z_{66} , and then the factor from the same block row as z_{66} ; it is z_{52} . Thus we add to \mathcal{P}_σ the route $(\hat{X}_{21}, X_{23}, \hat{X}_{33}, X_{31})$. Finally, $\mathcal{P}_\sigma = \{(\hat{X}_{12}, X_{12}), (\hat{X}_{21}, X_{23}, \hat{X}_{33}, X_{31})\}$

Observe that different permutations may correspond to the same associated route sets. Namely, for $k = 1$ there are two permutations and only one route. We say that two route sets \mathcal{P} and \mathcal{S} are equivalent if their elements are pairwise equivalent. Now, take a representative from each equivalence class of route sets received by our construction; denote the collection by \mathcal{P}_k .

4. THE DETERMINANT OF A 2-BLOCK MATRIX

Define the *length function* $l(P)$ as the number of factors of P , and the *index* given by

$$\nu(P) = \begin{cases} 1, & \text{if } P \text{ is of form } (\hat{X}_{ij}, X_{ij}) \text{ or } (X_{ij}, \hat{X}_{ij}), \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 1. Assume that $Z = (X_{rs})_{r,s=1}^k \in \text{Mat}_{2k \times 2k}(\mathbb{k})$ is a 2-block matrix, i.e., it is divided into blocks X_{rs} of size 2×2 . Then the following equality holds:

$$|Z| = \sum_{P \in \mathcal{P}_k} \prod_{P \in \mathcal{P}} (-1)^{\frac{1}{2}l(P)-1} 2^{-\nu(P)} \text{tr } P. \quad (4)$$

Proof. Both sides of (4) there are polynomials in matrix elements of X_{ij} . We claim that each z_σ from the left side occurs in the right side with the same coefficient.

Lemma 2. The product $\prod_{P \in \mathcal{P}} 2^{-\nu(P)} \text{tr } P$ equals a sum of monomials $\pm z_\sigma$, where σ run through such permutations that every $z_{p, \sigma(p)}$ lies in an element of some route $P \in \mathcal{P}$.

Proof. First prove the lemma for $\mathcal{P} = \{P\}$, $\nu(P) = 0$.

Note that a transposition of block rows or block columns changes neither $|Z|$, nor the right side of (4). Hence we may assume that $P = (\hat{X}_{11}, X_{12}, \hat{X}_{22}, X_{23}, \dots, \hat{X}_{kk}, X_{k1})$.

The polynomial $\text{tr } P$ is a sum of monomials in matrix elements of blocks X_{ij} of the following form

$$(\hat{X}_{11})_{p_1 p_2} (X_{12})_{p_2 p_3} (\hat{X}_{22})_{p_3 p_4} (X_{23})_{p_4 p_5} \dots (\hat{X}_{k,k})_{p_{2k-1} p_{2k}} (X_{k,1})_{p_{2k} p_1}. \quad (5)$$

Such a product equals some z_σ whenever each pair of its factors does not lie in one row or in one column. By construction of P only some $(X_{q,q+1})_{p_{2q} p_{2q+1}}$ and $(\hat{X}_{q+1,q+1})_{p_{2q+1} p_{2q+2}} = (X_{q+1,q+1})_{\bar{p}, \bar{q}}$

may lie in the same block column. They are in the same column if and only if $\tilde{q} = 2q + 1$. But for $X, Y \in \text{Mat}_{2 \times 2}(\mathbb{k})$ we have

$$X\hat{Y} = \begin{pmatrix} x_{11}y_{22} - x_{12}y_{21} & -x_{11}y_{12} + x_{12}y_{11} \\ x_{21}y_{22} - x_{22}y_{21} & -x_{21}y_{12} + x_{22}y_{11} \end{pmatrix},$$

and $b \neq d$ holds in all the products $x_{ab}y_{cd}$ that occur here. The same argument may be used to prove that every two factors of (5) do not lie in the same row.

If $\nu(P) = 1$, i.e., $P = (\hat{X}_{rs}, X_{rs})$, we have $\text{tr } P = 2|X_{rs}|$. This gives rise to the factors $2^{-\nu(P)}$ in (4).

Assume that there is more than one route in \mathcal{P} . Each route $\mathcal{P} = \{P_1, \dots, P_m\}$ determines a collection of block submatrices Z_{P_1}, \dots, Z_{P_m} such that each P_i is entirely situated in Z_{P_i} and each block row and each block column of Z intersects with a unique Z_{P_i} . By transpositions of block rows and columns we can make all the blocks Z_{P_i} diagonal. Now it is evident that $\prod_{P \in \mathcal{P}} 2^{-\nu(P)} \text{tr } P$ equals a sum of products $z_{\sigma_1} \dots z_{\sigma_m}$, where z_{σ_q} are summands of $|Z_{P_q}|$ and σ_j is associated to P_j . \square

Thus the right side of (4) is a linear combination of z_σ .

Lemma 3. *Each monomial z_σ occurs in $\prod_{P \in \mathcal{P}_\sigma} 2^{-\nu(P)} \text{tr } P$ with nonzero coefficient.*

Proof. Without loss of generality, we may assume that $\mathcal{P}_\sigma = \{P\}$ with $P = (\hat{X}_{r_1 s_1}, X_{r_1 s_2}, \hat{X}_{r_3 s_2}, \dots, X_{r_{2k} s_1})$. Denote by $z_{\sigma, q}$ the factor of z_σ lying in the q -th element of P . Fixing the row containing $z_{\sigma, 1}$, we determine the row, in which $z_{\sigma, 2}$ lies: it is the second row of a pair $(2r_1 - 1, 2r_1)$. Similarly, the choice of the column containing $z_{\sigma, 2}$ determines in which column $z_{\sigma, 3}$ lies, and so on. Finally, knowing the column containing $z_{\sigma, 2k}$ we learn in which column $z_{\sigma, 1}$ lies. Consequently, all the permutations τ with $\mathcal{P}_\tau = \mathcal{P}_\sigma$ are parametrised by tuples $\xi(\tau) = (\xi_1, \dots, \xi_k) \in (\mathbb{Z}_2)^{2k}$. Further, if we interchange the $(2r_1 - 1)$ -th and the $2r_1$ -th rows of Z , then ξ_1 becomes $1 - \xi_1$ and the other elements of a tuple do not change. Thus there exists a sequence of such permutations sending a product z_σ to any z_τ with the same associated set. It is only left to understand, how these transformations change $\text{tr } P$.

If we interchange the $(2s_q - 1)$ -th and the $2s_q$ -th columns, it only influences the fragment $(X_{r_{q-2} s_q}, \hat{X}_{r_q s_q})$. But for $X, Y \in \text{Mat}_{2 \times 2}(\mathbb{k})$ we have

$$X\hat{Y} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} y_{22} & -y_{12} \\ -y_{21} & y_{11} \end{pmatrix} = \begin{pmatrix} x_{11}y_{22} - x_{12}y_{21} & -x_{11}y_{12} + x_{12}y_{11} \\ x_{21}y_{22} - x_{22}y_{21} & -x_{21}y_{12} + x_{22}y_{11} \end{pmatrix},$$

and after the permutation:

$$\begin{pmatrix} x_{12} & x_{11} \\ x_{22} & x_{21} \end{pmatrix} \begin{pmatrix} y_{12} & y_{11} \\ y_{22} & y_{21} \end{pmatrix} = \begin{pmatrix} x_{12}y_{21} - x_{11}y_{22} & -x_{12}y_{11} + x_{11}y_{12} \\ x_{22}y_{21} - x_{21}y_{22} & -x_{22}y_{11} + x_{21}y_{12} \end{pmatrix}.$$

Comparing the results we see that both products contain the same members, yet with different signs. So a transposition multiplies $\text{tr } P$ by (-1) . The same argument works for transformations of columns. Hence all z_τ with $\mathcal{P}_\tau = \{P\}$ occur in $\text{tr } P$ with nonzero coefficient, which in fact equals ± 1 . \square

It is clear that the same z_σ can not occur as a summand in $\prod_{S \in \mathcal{S}} 2^{-\nu(S)} \text{tr } S$ and $\prod_{P \in \mathcal{P}} 2^{-\nu(P)} \text{tr } P$ for two different sets \mathcal{P} and \mathcal{S} in \mathcal{P}_k . Therefore each z_σ occurs in both sides of (4) with coefficient ± 1 . It remains to prove that these signs are the same.

Assume as before that $\mathcal{P}_\sigma = P$, where $P = \hat{X}_{11}X_{12}\hat{X}_{22} \dots X_{l(P), 1}$. We know that if $\mathcal{P}_\tau = \mathcal{P}_\sigma$, then there exists a permutation of rows and columns of Z multiplying both sides of (4) by the same number 1 or (-1) and transforming z_σ into z_τ . So the question is whether the coefficients of a fixed z_{σ_0} are the same. Consider $\sigma_0 = (1, 2, 3, \dots, 2l(P) - 1, 2l(P))$. Then the coefficient of z_{σ_0} in $|Z|$ equals $(-1)^{\sigma_0} = -1$. As for the right side, multiply the summand

$$\pm z_{12}z_{23} \dots z_{2l(P)-1, 2l(P)} z_{2l(P), 1} = (\hat{X}_{11})_{21}(X_{12})_{21}(\hat{X}_{22})_{21}(X_{23})_{21} \dots (\hat{X}_{l(P), l(P)})_{21}(X_{l(P), 1})_{21}$$

by $(-1)^{\frac{1}{2}l(P)-1}$. We have $(X_{q, q+1})_{21} = z_{2q, 2q+1}$ and $(\hat{X}_{qq})_{21} = -z_{2q-1, 2q}$, so the total sign is $(-1)^{\frac{1}{2}l(P)-1}(-1)^{\frac{1}{2}l(P)} = -1$. This completes the proof of Proposition 1. \square

5. PROOF OF THEOREM 2

Recall that by Theorem 3 the algebra $\mathbb{k}[\text{Rep}(Q, \alpha)]^{SL(\alpha)}$ is generated by the coefficients of monomials in y_{rs} in determinants of block matrices of form (3). So it is only left to prove that these are precisely the traces of routes associated to those matrices.

It is easy to see that in the determinant of $(y_{rs}X_{rs})_{r,s=1}^k$ the coefficient of $y_{r_1s_1} \dots y_{r_ks_k}$ equals the alternating sum of all such z_σ that there exists a $\tau \in S_{2k}$, for which every $z_{p\sigma(p)}$ is in $X_{r_{\tau(p)}s_{\tau(p)}}$. Observe that each block row and each block column contains precisely two blocks $X_{r_p s_p}$ and hence these blocks are elements of a route from some set \mathcal{P} . Now from the proof of Proposition 1 it is clear that the coefficient of $y_{r_1s_1} \dots y_{r_ks_k}$ equals $\prod_{P \in \mathcal{P}} (-1)^{\frac{1}{2}l(P)-1} 2^{-\nu(P)} \text{tr } P$.

Now prove that traces of the routes under consideration are semi-invariants. If $\hat{X}_1, \dots, \hat{X}_N$ are all the adjoint factors of a route P (with multiplicities), then

$$\text{tr } P = \text{tr } \hat{X}_1 P_1 \dots \hat{X}_N P_N = |X_1| \cdot \dots \cdot |X_N| \text{tr } X_1^{-1} P_1 \dots X_N^{-1} P_N,$$

where P_i are products of the remaining elements of P . Since $\text{tr } X_1^{-1} P_1 \dots X_N^{-1} P_N$ is a rational invariant, $\text{tr } P$ and $|X_1| \cdot \dots \cdot |X_N|$ have the same weight.

Further, if a factor X appears twice in a route P , then by Lemma 1 we have

$$\text{tr } P = \text{tr } X P_1 X P_2 = \text{tr } X P_1 \text{tr } X P_2 - \frac{1}{2} \text{tr } \hat{X} X \text{tr } P_1 \hat{P}_2.$$

The sequence (P_1, \hat{P}_2) is a route yielding that $\text{tr } P_1 \hat{P}_2$ is a semi-invariant. So the algebra of semi-invariants is generated by traces of simple routes. Theorem 2 is proved.

Observe that the trace of a route containing a pair $\hat{\varphi}_a, \hat{\varphi}_b$ with $ha = tb, ta = hb$ may be excluded from the generating set. Indeed, such a route is of form $(\hat{\varphi}_a, P, \hat{\varphi}_b, S)$, where P and S are subroutes. But for $X, Y \in \text{Mat}_{2 \times 2}(\mathbb{k})$ Lemma 1 implies that $|X| \text{tr } Y \hat{Z} = \text{tr } XY \text{tr } XZ - \text{tr } XYXZ$. Therefore,

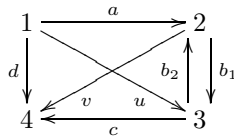
$$\begin{aligned} \text{tr } \hat{\varphi}_a P \hat{\varphi}_b S &= \frac{1}{|\varphi_a|} |\varphi_a| \text{tr } \hat{\varphi}_a P \hat{S} \varphi_b = \frac{1}{|\varphi_a|} (\text{tr } \varphi_a \hat{\varphi}_a P \text{tr } \varphi_a \hat{S} \varphi_b - \text{tr } \varphi_a \hat{\varphi}_a P \varphi_a \hat{S} \varphi_b) = \\ &= \frac{1}{|\varphi_a|} (|\varphi_a| \text{tr } P (\text{tr } \varphi_a \varphi_b \text{tr } S - \text{tr } \varphi_b \varphi_a S) - |\varphi_a| (\text{tr } \varphi_b P \varphi_a \text{tr } S - \text{tr } \varphi_b P \varphi_a S)) = \\ &= \text{tr } \varphi_a \varphi_b \text{tr } P \text{tr } S - \text{tr } P \text{tr } \varphi_b \varphi_a S - \text{tr } S \text{tr } \varphi_a \varphi_b P + \text{tr } \varphi_b P \varphi_a S. \end{aligned}$$

If the initial product $\hat{\varphi}_a P \hat{\varphi}_b S$ were assigned to a route, then P and S should map from W_{ha} to W_{ha} and from W_{ta} to W_{ta} respectively:

$$W_{ta} \xleftarrow{\hat{\varphi}_a} W_{ha} \xleftarrow{P} W_{ha} \xleftarrow{\hat{\varphi}_b} W_{ta} \xrightarrow{S} W_{ta}.$$

So the products whose traces we take in the final formula are also associated to routes.

Finally, note that for 3-representations Theorem 2 does not hold. Indeed, consider the quiver



and the dimension vector $(3, 3, 3, 3)$. The coefficient F of $\prod_{i,j} y_{ij}$ in the determinant of the block matrix

$$Z = \begin{pmatrix} y_{11}\varphi_a & y_{12}E & y_{13}\varphi_{b_2} \\ y_{21}\varphi_u & y_{22}\varphi_{b_1} & y_{23}E \\ y_{31}\varphi_d & y_{32}\varphi_v & y_{33}\varphi_c \end{pmatrix}$$

is not invariant, and its total degree with respect to the variables of each φ equals 1. On the other hand, the trace of a route is not invariant if and only if the route contains an adjoint matrix. The total degree of a trace with respect to the variables from this matrix is at least 2. Thus the subalgebra generated by the traces of routes does not contain F .

REFERENCES

- [1] *Arzhantsev I.V., Hausen J.* Geometric Invariant Theory via Cox rings. — J. Pure Appl. Algebra **213:1** (2009), 154-172
- [2] *Derksen H., Weyman J.* Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients. — J. Amer. Math. Soc. **13:3** (2000), 467-479
- [3] *Domokos M., Zubkov A.N.* Semi-invariants of quivers as determinants. — Transformation Groups **6:1** (2001), 9-24
- [4] *King A.D.* Moduli of representations of finite-dimensional algebras. — Quart. J. Math. Oxford Ser. (2) **45(180)** (1994), 515-530
- [5] *Procesi C.* The invariant theory of $n \times n$ matrices. — Adv. Math. **19** (1976) 306-381
- [6] *Razmyslov Y.P.* Trace identities of full matrix algebras over a field of characteristic zero. — Math. USSR, Izv. **8** (1974), 727-760
- [7] *Schofield A., Van den Bergh M.* Semi-invariants of quivers for arbitrary dimension vectors. — Indag. Math. (N.S.) **12** (2001), 125-138

117463, NOVOYASENEVSKY PR-T, 32-1-559, MOSCOW, RUSSIA
 MOSCOW STATE UNIVERSITY, DEPARTMENT OF HIGHER ALGEBRA
 EMAIL: GLWRATH@YANDEX.RU